THIRD EDITION

Physics Newsletter

DEPARTMENT OF PHYSICS
IIT Bombay

Welcome reader,

to the third edition of our newsletter, and the first instance of one beginning with an editorial.

In a nut ‘graph, our newsletter is envisioned as an outlet for science communication from the department’s various representatives and stakeholders. Being one among the umpteen student undertakings in the institute, the underlying value of a department newsletter being a manifesto point often ends up overshadowing its conceptual worth, but you’ll find undeniably honest and informative pieces contained within. Every year we reach out to students, alumni, and faculty to pen down a few paragraphs that have the thread of physics and the department running through them; this year’s yield is packaged herein.

This edition marks the Diamond Jubilee of the institute, and that of the department too, since ours was one of the only two science departments set up when IIT Bombay had just begun its material existence. The trajectory of this department has been predictably chaotic, with the exception of year 2018 when the anti-infi-corridor gate of the department became operational—none of us saw that coming. There’s nobody better to explore this journey with than a budding Condensed Matter Physicist; head over to Arkya’s article to know more about how the department evolved out of the Y2K bug, and the tumbling dominoes that led to the current 1:1 ratio of experimentalists versus theorists among the faculty.

Incidentally, I happened to interact with the Bangalore science community over this summer, and found out to my amazement that a course named Engineering Physics still results in puzzled looks and my explanation (“three years of B.Sc. Physics with a bit of Electronics sprinkled in”) doesn’t help much. I had no more than a fuzzy idea of this branch when I started, and as it turns out, so did every other student of the course. Relatively fresh alumni Sandesh and Ayush, a graduate student and a financial analyst, respectively, discuss how they dealt with this uncertainty in their own ways. Their quest to find answers brings them to pen hauntingly nostalgic notes on how the department becomes a home for so many students stepping away from parental love and protection for the sake of their love of physics.
While Sandesh and Ayush maneuver the difficulties in defining an education in physics, Viraj, on the other hand, grapples with defining an orientation to research in physics. In “Width or Depth?”, Viraj weighs narrowing down on a topic versus venturing forth into the broad horizons; he delineates how his current research interest is a mix of happy accidents, and ends on a reassuring note to the current undergrads.

The newsletter takes on a rigour of its own with Reebhu’s piece on Differential forms, which itself is a continuation of his article on Tensors from last year. At this point it will be beneficial to add a confessory note from my side acknowledging that my editorial prerogative ended at ensuring literary accessibility of Reebhu’s article, and that I was as much a novice at Differential forms as any recent physics graduate might be.

Finally, we have an attractive jacket and accompanying illustrations from Parimal, another alumnus, who chose neither the research path nor the industry-finance vortex after graduating. We attribute, not inaccurately, the majority of perusals that our magazines and newsletters attract to Parimal’s own style of intuitive and refreshing designs.

I hope you find something worth keeping from this edition.

-Toshi Parmar

Powai Perspectives on Physics

- Prof. K. G. Suresh

A Message from the HoD on Dept. of Physics' Celebration of the Diamond Jubilee

Prof. K. G. Suresh has been the Head of the Department of Physics since November 2017.

As part of the diamond jubilee celebrations of IIT Bombay, the department of Physics organized a 2-day research conclave titled "Powai Perspectives on Physics" during October 12 and 13, 2018. Leading researchers, comprising of some of our alumni, gave lectures on various topics ranging from topological materials to gravitational waves to memory and plasticity. Eminent scientists from a few institutes across the country also gave lectures. Selected PhD students from the department also gave talks on different topics. It was attended by a large number of students at all levels, cutting across departments of IITB as well as from colleges/institutes in Mumbai.

The event gave an opportunity for an interaction between the existing faculty, retired faculty and alumni. All the participants stressed on the need and utility of establishing long term research collaborations with the department.

Physics @ IITB Through the Ages

- Arkya Chatterjee

Arkya Chatterjee is from the graduated batch of 2019, and is currently experiencing the debilitating realities of adulthood such as finding a plumber in Boston. He is the ex-DAMP head for Physics department, an avid listener of rock, and for the rest you can just head up to his CV.

1. http://www.hss.iitb.ac.in/en/history-department
2. http://www.iitb.ac.in/en/about-iit-bombay/institute-history
3. https://powai.info/2010/04/02/history-of-iit-bombay-video/
4. http://www.phy.iitb.ac.in/en/about-us/history

IIT Bombay was established in 1958, as the second IIT in the country after Kharagpur, with five engineering departments, two science departments and a department of languages[1]. The institute was set up in cooperation with UNESCO, and with substantial infrastructural assistance and expert services from the erstwhile USSR during the first couple of decades[2]. **Physics was, in fact, one of the first departments that the institute started off with.** In the first year of operation, IIT Bombay admitted around 100 students from an applicant pool of approximately 3400[3]. Today, 60 years later, the institute admits a little more than 1000 students in undergraduate (UG) programs via the nation-wide JEE examinations that attract nearly 1.2 million candidates. Over the years, as the student intake increased, the academic programs of the department went through a number of revisions and amendments under the guidance of its many able-minded leaders. Alongside those changes the research outlook and focus have also seen a continuous but highly exploratory trajectory. We aim to uncover a bit of that history in this Diamond Jubilee edition article.

The Physics department started off offering courses in Chemical Physics. Later on, a (post-BSc) MSc Physics degree was started. This was followed by a brief period from 1972-1983, in which a 5-year integrated MSc degree in Physics was offered. **With the switch from the 5-year BTech program to a 4-year one, the appeal for the integrated MSc program diminished, leading to its termination in 1983. The termination of the integrated MSc program paved the way for the 4-year Engineering Physics which continues to this day[4].** In its first few batches, the number of students was typically a single-digit figure.

Over the years, the program strength has more than quadrupled to its current strength of 44.[5] Almost an equal number of students are admitted into the 2-year MSc program. Ever since the early 2000s, the Physics department also has had an active PhD program with around 150 research scholars.

What is history but a survivors' tale? We uncover the intractable timeline of the department's history through the accounts of the department's own people.

Prof. S.H. Patil is one of the seniormost faculty still associated with the department of Physics. Currently an emeritus professor, he still visits his office almost every single day and jokes around with students whenever he sees them! In one such informal session, we found out that he joined the department around 1968 and was a professor here until his retirement in 2010. He recounts how, in the good ol' days, all the faculty members took a general interest in each other's work, so much so that whenever someone published a paper, there would be a general interest in discussing the work among the faculty. Over the years, he feels that the number of sub-disciplines of research have spread out quite a bit, leading to a decrease in this kind of interaction among the various faculty members.

The diversification of the department's research thrust didn't happen easily though. First, up until the late nineties, IIT Bombay used to be more of a teaching institute than a research one. This was the main reason why the PhD program was rather "perfunctory", as one of the senior professors describes. Moreover, the faculty in the Physics department were majorly involved in theoretical work, with only a few experimental groups. **During the nineties, a particularly dry period in terms of funding, a number of new faculty started the culture of funding their research through grants.** These majorly included experimentalists such as Profs. P. Das, T. Kundu, B.P. Singh, R. Varma among others. Their efforts led to the start of a culture of independent research funding. Since then, funding for experimentalists have increased significantly. In lockstep with funding, the number of experimental researchers has gone up. In stark contrast with the early years of the department, the ratio of experimentalists to theorists among the current faculty is almost exactly 1:1.[6]

Prof. Urjit Yajnik (class of '80) jokingly claims to belong to "the rare species" of integrated MSc students of the Physics department. He joined his alma mater as a professor in 1989 and has been associated with it ever since. He was quite popular as Dean of Student Affairs (DoSA) during 2011-15 and continues to be a favourite among students. His long association with the department meant that he would be a goldmine of information. We interviewed him and definitely weren't disappointed! Prof. Yajnik's early years as a professor in the nineties were marked by a general lack of any major central funding since the government had decided that IITs should be independent in funding their research. At the same time, the number of PhD students (not just in Physics but across all departments) also dwindled due to disproportionately low stipends in IIT Bombay as opposed to other top institutes in the country. In the 2000s, however, "possibly due to the Y2K bug" quips Prof. Yajnik, the number of research scholars started rising, and both the quality and quantity of student intake in the PhD program improved significantly.

5. http://www.iitb.ac.in/newacadhome/

6. http://www.phy.iitb.ac.in/en/faculty

Prof. Avinash Mahajan (class of '86), belonging to the very first batch of EP students , recalls taking electronics courses alongside Electrical Engineering students since the electronics teaching labs were not well developed yet. Moreover, the course structure in the early years was quite rigid with very few electives, and no option of doing a minor, and so on. **He recalls a significant increase in the freedom to take up electives post the Biswas committee revision in the mid 2000s.** He was able to shed some light on the way the department's research program is planned. The Department Policy Committee, in consultation with the faculty body, identifies certain thrust areas in which faculty hiring is concentrated. This body has been instrumental in guiding the department's research over all these years. Especially in the 2000s, when biophysics and nonequilibrium statistical mechanics were becoming prominent, the department started making a number of hires in this area, leading to the formation of the current biophysics and soft matter group. However, beyond faculty hirings in specific areas, the department doesn't really get involved in drafting plans for existing faculty, who are given more or less a free rein on whatever they wish to work on. Still, there are government incentives for certain research areas that have changed with time, e.g. the mission mode projects on laser systems in the 80s and 90s[7] and the current thrust in quantum computing[8].

Coming to pedagogy and faculty-student interaction, almost all of the professors we spoke with - most of them have been students in the department once - agreed that the core interaction and teaching methodology has not changed too much. The primary reason behind this is that, since Physics batches (both BTech and MSc) are relatively small, classic blackboard teaching is still preserved in almost all courses. This gives students the chance to engage in discussions more effectively. We spoke with **Prof. Archana Pai** who completed her MSc degree in Physics in our department in 1996. Having spent her early career years as a professor at IISER Trivandrum, she joined her alma mater in 2017. Being one of the most friendly professors in the department, she was able to give us the heartening feedback that students seem to be much more cooperative now than they used to be. She notes, however, that back in her day, all core courses used to take place in classrooms located inside the department which meant they could see the teachers around almost all day. Nowadays, since classes mostly take place in the Lecture Hall Complex, the chance to interact with professors has gone down. But she does admit that the DAMP team's efforts have helped in bridging the communication gap to some extent.

Somewhat in sync with the Diamond Jubilee, in the past couple of years we saw a number of alumni coming back to the department as professors. In most reputed universities around the world, this is seen as a sign of the growing reputation of research culture and academic community. Of course, our institute and our department have always enjoyed a position of prestige in academic circles both in India and in the world at large. With the Diamond Jubilee celebrations coming to a close, we students hope that this upward sloping trajectory doesn't face a point of inflection.

Acknowledgement:
The author would like to thank Professors Archana Pai, Avinash Mahajan, S.H. Patil, and Urjit Yajnik for sharing their experiences.

7. https://www.iitk.ac.in/new/data/History-of-R-D-2April2009.pdf
8. https://thewire.in/the-sciences/govt-promises-shot-in-the-arm-for-quantum-tech-research-in-india

Engineer?
Physicist?
err...
Engineering Physicist?

- Kumar Ayush and Sandesh Kalantre

Ayush is a financial analyst at Goldman Sachs, Bengaluru. Sandesh is a graduate research assistant in condensed matter physics at the University of Maryland, College Park. They both received their undergraduate degree in Engineering Physics (class of 2018) at IIT Bombay. This article is a personal memoir of their beloved department, their life as EP students and everything around it.

Engineer? Physicist? Err… Engineering Physicist? In hindsight, it would be an understatement to say that the past four years have been more about discovering ourselves than anything else. Given the humdrum reality of scientifically inclined teenagers in India, it was no wonder, we both ended up being at an engineering institution, for us, that being our beloved 'insti', IIT Bombay. Nevertheless, it was the Engineering Physics program that got us the best of both worlds, the security of much sought after a degree from an IIT and a way to satiate our curiosity in physics and mathematics.

We were both attracted to this strangely sounding programme at IIT Bombay because of our love for physics and mathematics from high school. For Ayush, it was "partly because Prof. Varun Bhalerao said it'd be a good idea, partly because there was some family pressure to join an IIT rather than IISc which was my next best option, and partly because Sandesh too chose IITB". **It's probably incorrect to listen to others or do as your friends are doing, but if you've chosen the people and the friends well, it's not as bad as many make it sound.**

Education in physics is a strange thing to define. It's challenging to draw borders in what constitutes physics and what does not. The study of stars, galaxies and black holes does belong to the realm of physics, but so does getting an operational amplifier to work in a lab. Statistical physics of gases is used to analyze the workings of earthly things like metals or superconductors to astronomical objects like white dwarfs. But more crucially so, the same ideas show up in the study of financial markets. The brilliant platter of physics courses at IITB got to us the nitty gritty details of all the above ideas. But more so, it was the wisdom about the interconnections of science and an appreciation of all things, physics or not, that we learned during our stay here stays with us today.

Receiving education as an undergraduate is never a solitary activity. IITB is no exception. The general idea was that IITB would offer a better and varied peer which is important for undergraduate education. We were not just joining to learn Physics. We were joining to learn how to live a life. And the variety along with the intensity of experiences matters a lot for the latter. Did we know all of this going in, or did it come with four years of retrospect? Or something in between like for Ayush - “I don’t know, I have a bad memory but if I were to guess I’d say yes I did know this going in”.

Coming from Pune and Jodhpur to Mumbai meant we were going to away from a homely environment. **Though we were no longer at home, a new home itself came to us as we progressed through the years, the department.** We recall a department introduction session for freshmen, which involved professors giving short descriptions of their work. We started with the senior most professors first, and as we progressed through the introductions, a happy coincidence occurred. The younger professors were, in fact, students of the older ones. After three or so such generations, came the youngest freshmen in the room, the actual students. The department constituted of a significant fraction of such alumni professors, which made being associated with the place quite rewarding. They had a sense of gratitude towards it and an anchor to their past when they were students. That’s what made them so relatable to us and vice versa. A multitude of interactions with them made us into members of a very ‘homely’ tradition, which we hope to keep up where we go.

Categorizing the physics department as a home is incomplete without talking about the computer lab and the department library. These two places have been fortuitously available for use to all department students during recent years. Open 24x7, they often vacillated between the hustle and bustle of a noisy cafe to a place of necessary solitude for academic work. **Well, what makes a college a good college is its collegiality. We spent many weekend mornings there, over coffee and an assorted set of crazy ideas about physics or everything else.** Ayush spent many sleepless nights doing hackathons, completing reports and so on. The library also doubled up as a visitor office which meant we could meet other students and professors. These rooms brought people and ideas together and they continue to do so today.

At the end of our stay here, we diverged onto two different paths - as a financial analyst and as a graduate student in physics. For Ayush, he “never understood why people call my job as the ‘non-core’ sector. Broadly speaking, EP has partly applied math and experimental physics. Sure, I don’t use the latter in my job, but I use or see a use for most of the other skills that I learnt in the former. The interesting part of studying nature was the intellectual process. In physics, the fight is against the rules of nature, while in finance it is against the rules of humans. It is this challenge that excites me to work, every day.” As for Sandesh, “it was the same love for physics that brought me at IITB takes me to the lab almost every day as a graduate student. In our respective jobs, we still do the same thing we were taught so well at IITB, we make models and solve problems - either on exotic superconductors or the financial market is a matter of taste and details."

As we look forward to our jobs for tomorrow - either measure a sample at ultra-low temperatures or use the Black-Scholes pricing model, all of it would be impossible without our education as an Engineer and as a Physicist and as much more during our stay here. Hindsight is always 20/20, and getting an EP degree was the best thing to do. If we could rewind 4 years and could choose a different route, would we? Absolutely for one, and even perhaps do it once more so.

Viraj Karambelkar is from the graduated batch of 2019, and he's the ex-SAPD GSec for the department. An avid astro-enthusiast turned graduate astrophysics student, Viraj has a passion for plays - whether it be acting in one, or writing it. Viraj also boasts of a marvelous lung capacity, and he's utilised it to be a flautist.

Two years ago, at the end of my sophomore year, I was hanging out with a friend in Mumbai. He casually asked me "What work would you like to do after graduation?". My standard response to this was "Not yet decided, but some research in quantum computing or astrophysics would be pretty cool". I asked him the same question expecting a similar answer, but he said "I want to do a PhD on gauge asymmetries in type p bosons." More than half of his words were out of my vocabulary, which did not surprise me as my general knowledge of particle physics is not very profound. What surprised me was the specificity of his answer. The disparity in both our answers made me a bit insecure, as it made me wonder whether I too needed to have such a clear idea about my future. Thankfully, none of my friends from IIT had such specific plans, which put me at ease.

However, this did leave me thinking : **what is better - to have a clear idea of what niche you wanted to work in at an early stage in your academic life, or delay this decision as much as possible?** It made me reflect on my motivation to pursue research. In school, I always wanted to do something related to astronomy. This was what drove me to choose engineering physics, as it could possibly lead to an entry in astrophysics. At IIT, however, there was not much that a largely ignorant first year student could do in terms of research in astronomy, or for that matter, research in any field. In this situation, I jumped at the first chance of doing a research project in any field. I had the chance to attend a workshop conducted by NIUS to inculcate first year undergrads into research. This workshop had a series of lectures on different fields in physics such as particle physics, astrophysics, experimental physics and quantum computing. At the end of the series, the speakers offered research projects to the attendees, and students were selected on the basis of a test. At the beginning, I had made up my mind to try for an astrophysics research project. But I was blown away by the lecture on quantum computing. This speaker intrigued me by demonstrating how simple principles of quantum mechanics coupled elegantly with linear algebra could be used to formalise complex processes such as teleportation. Consequently, I tried hard on the test and got accepted to work on a research project in quantum computation at IISER Kolkata. As my first research experience, I worked on trying to quantify some theoretical aspects of quantum entanglement. **I stayed in IISER Kolkata for three weeks, and I had made up my mind to study quantum information for the rest of my life.** I continued to work on this project for some time in my fourth semester, but the pace of work gradually slowed down as coursework increased and my advisor too became busier. Eventually, I stopped working , but still planned to renew this project in vacations.

At this time, as part of an electronics lab project, Prof. Sarin encouraged us to talk to a new professor who had joined the department and was an IITB electrical engineering alumnus. With my project partner, I met with Prof. Bhalerao to discuss possible FPGA projects. And I found out that he actually worked in astronomy! I found the kind of work he described fascinating, and a welcome opportunity to rekindle my interest in astronomy. I started working on a long-term astronomy project with him, which led to an amazing summer research internship. I started working in this field through an accident, I have now decided to pursue research in astronomy!
So what do students from other institutes think about exploring? I got to spend a summer abroad, where I met many undergraduate students from different countries. **What struck me was the variety of things that these students did.**
One student from Singapore was an Economics major, who was doing research in Astronomy and had already published a paper. He planned to go back and pursue his interest in Economics. A student majoring in computer science had summer plans to teach English in China. An astronomy workshop had attendees from a diverse background of majors, all the way from Philosophy to Computer Science. Granted that the education systems differ from place to place, a considerable fraction of students that I came across in my short stay seemed willing to explore very diverse avenues in their undergraduate years.

So which is the right way? Explore a topic in full depth or explore a wide range of topics during undergrad? Of course the answer is subjective.
I know friends who have done both - someone who stuck to a topic and studied it in great depth, and someone who has explored diverse topics like biology, astronomy and electronics in 4 years, and they are all doing perfectly fine in their careers. I am in no position to justify one option over the other. **Had I not walked into a professor's office some years ago, I would still be doing quantum computation with equal passion.** At the end of the day, you have to figure out what works best for you. However, if you are amongst those who haven't figured out what to do yet and are looking to explore, I can assure you that there are loads of students worldwide who do the same. So do not be discouraged from exploring as much as you can.
Undergrad happens only once.

Differential Forms: Beyond Tensors

- Reebhu Bhattacharya

Reebhu Bhattacharya is from the graduated batch of 2019. A recipient of the highly elusive Integrated Masters degree in Mathematics, Reebhu started his journey as an undergrad in the Physics department.

In the previous edition of the newsletter, we had covered the basic mathematical definition of tensors and the tensor product. In this article, we go one step ahead and deal with differential forms and tensor fields. As a small reward for our efforts, we will see how this helps us state the Maxwell equations in an elegant form.

Recall, from the previous article, the tensor product of two vector spaces $\mathbb{V}$ and $\mathbb{W}$ was defined to be a quotient of the free vector space on $\mathbb{V} \times \mathbb{W}$. For the purpose of this article, we can more simply think of the tensor product $\mathbb{V} \otimes \mathbb{W}$ as the vector space spanned by linearly independent elements $\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ where $\{v_i\}_{i=1}^n$ and $\{w_j\}_{j=1}^m$ are chosen fixed bases of $\mathbb{V}$ and $\mathbb{W}$ respectively.

Recall also, that for a vector space $\mathbb{V}$, $\mathbb{V}^*$ denotes its dual vector space. The tensors of rank (r, s) on $\mathbb{V}$ form a vector space $\mathcal{T}^r_s(\mathbb{V})$, which can be identified with $T^r(\mathbb{V}) \otimes T^s(\mathbb{V}^*) = \mathbb{V}^{\otimes r} \otimes (\mathbb{V}^*)^{\otimes s}$.

1 Symmetric and Alternating Tensors

Let $\alpha \in T^k(\mathbb{V}^*) = \mathbb{V}^{*\otimes k}$ be a covariant k-tensor over $\mathbb{V}$. We can equivalently view α as a multilinear map $\mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{R}$, namely if $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_k$ for $\alpha_j \in \mathbb{V}^*$, then for $v_i \in \mathbb{V}$, $\alpha(v_1, \cdots, v_k) = \alpha_1(v_1) \cdots \alpha_k(v_k)$. Let $\mathfrak{S}_k$ denote the group of permutations of k symbols, the k-th symmetric group.

Note that $\mathfrak{S}_k$ acts on V^k by $\sigma(v_1, \cdots, v_k) = (v_{\sigma^{-1}(1)}, \cdots, v_{\sigma^{-1}(k)})$, for $\sigma \in \mathfrak{S}_k$. Also, $\mathfrak{S}_k$ acts on the space of covariant k-tensors by

$$\begin{aligned} {}^{\sigma}\alpha(v_1, \cdots, v_k) &= \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= \alpha(\sigma^{-1}.(v_1, \cdots, v_k)) \end{aligned}$$

Note that for $\tau, \sigma \in \mathfrak{S}_k$, ${}^{\tau\sigma}\alpha = {}^{\tau}({}^{\sigma}\alpha)$. Also observe, ${}^{\sigma}\alpha(\sigma.(v_1, \cdots, v_k)) = \alpha(v_1, \cdots, v_k)$.

We say that $\alpha \in T^k(\mathbb{V}^*)$ is *symmetric* if ${}^{\sigma}\alpha = \alpha$ for any $\sigma \in \mathfrak{S}_k$, or in other words, $\alpha(v_1, \cdots, v_k) = \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$ for any $\sigma \in \mathfrak{S}_k$. One can easily check that the set of symmetric covariant k-tensors forms a vector subspace of $T^k(\mathbb{V}^*)$, which we denote by $\Sigma^k(\mathbb{V}^*)$.

We define a projection Sym $: T^k(\mathbb{V}^*) \to \Sigma^k(\mathbb{V}^*)$ by

$$\text{Sym } \alpha = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} {}^{\sigma}\alpha$$

or, more explicitly,

$$(\text{Sym } \alpha)(v_1, \cdots, v_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

for $\alpha \in T^k(\mathbb{V}^*)$. It is a linear map and is called the symmetrisation operator. It is easy to verify that Sym (α) is indeed symmetric and that Sym $(\alpha) = \alpha$ if and only if α is symmetric.

The tensor product of a symmetric k-tensor and a symmetric l-tensor need not be a symmetric $k+l$-tensor. To rectify this, we introduce a new product called symmetric product. Let $\alpha \in \Sigma^k(\mathbb{V}^*)$ and $\beta \in \Sigma^l(\mathbb{V}^*)$. We define the symmetric product of α and β, denoted $\alpha \cdot \beta$ by

$$\alpha \cdot \beta := \text{Sym } (\alpha \otimes \beta) \in \Sigma^{k+l}(\mathbb{V}^*)$$

Then, defining $\Sigma(\mathbb{V}^*) = \bigoplus_k \Sigma^k(\mathbb{V}^*)$, we see that $\Sigma(\mathbb{V}^*)$ is an associative unital commutative algebra over $\mathbb{R}$. Identifying $\mathbb{V}$ with $\mathbb{V}^{**}$ and defining $\Sigma(\mathbb{V}) = \Sigma((\mathbb{V}^*)^*)$ as above, we obtain the algebra $\Sigma(\mathbb{V})$ which is called the symmetric algebra on V. Fixing a basis, $\{v_j\}_{j=1}^{j=n}$ of $\mathbb{V}$, one can check that $\Sigma(\mathbb{V})$ is isomorphic to the polynomial algebra in n variables, that is, $\Sigma(\mathbb{V}) \cong \mathbb{R}[v_1, \cdots, v_n]$.

We now provide a direct and more abstract construction of the symmetric algebra on $\mathbb{V}$. Consider the tensor algebra on $\mathbb{V}$, $T(\mathbb{V})$. Let I_Σ denote the ideal generated by elements of the form $x \otimes y - y \otimes x$ for $x, y \in \mathbb{V}$(this just means take the subspace spanned by all elements of the form $(\alpha \otimes x \otimes y \otimes \beta - \alpha \otimes y \otimes x \otimes \beta)$ where $\alpha, \beta \in T(\mathbb{V})$ and $x, y \in \mathbb{V}$). We can form the quotient $T(\mathbb{V})/I_\Sigma = \widetilde{\Sigma}(\mathbb{V})$. We denote the image of the tensor $\alpha \in T(\mathbb{V})$ in $\widetilde{\Sigma}(\mathbb{V})$ by $[\alpha]$. Note that the tensor product induces a product on this quotient, and this product is commutative unlike the tensor product. We, by abuse of terminology, also call this the symmetric product and the product of $\tilde{\alpha} = [\alpha]$ and $\tilde{\beta} = [\beta]$ in $\widetilde{\Sigma}(\mathbb{V})$ is given by $\tilde{\alpha} \cdot \tilde{\beta} = [\alpha \otimes \beta]$.

We have an explicit isomorphism between $\Sigma(\mathbb{V})$ and $\widetilde{\Sigma}(\mathbb{V})$. Consider the map Sym $: \mathbb{V} \to \Sigma(\mathbb{V})$, it is an algebra homomorphism, that is Sym $(\alpha \otimes \beta) =$ Sym $(\alpha) \cdot$ Sym (β). Its kernel contains I_Σ, so it induces an algebra homomorphism,

$$\varphi : \widetilde{\Sigma}(\mathbb{V}) = \mathbb{V}/I_\Sigma \to \Sigma(\mathbb{V})$$

It is obviously surjective, one can also check that it is injective so that φ is in fact an isomorphism.

The signed analogue of the above gives us alternating tensors. We say that $\alpha \in T^k(\mathbb{V}^*)$ is *alternating* if ${}^\sigma\alpha = (\text{sgn }\sigma)\alpha$ for any $\sigma \in \mathfrak{S}_k$ where sgn σ denotes the sign of the permuation σ. In other words, α is alternating if $\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = (\text{sgn }\sigma)\alpha(v_1, \cdots, v_k)$ for any $\sigma \in \mathfrak{S}_k$. Once again, the set of alternating (covariant) k-tensors forms a vector subspace of $T^k(\mathbb{V}^*)$, which we denote by $\bigwedge^k(\mathbb{V}^*)$.

We have a projection, Alt $: T^k(\mathbb{V}^*) \to \bigwedge^k(\mathbb{V}^*)$ given by

$$\text{Alt }(\beta) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}(\text{sgn }\sigma)\ {}^\sigma\beta$$

or equivalently,

$$(\text{Alt }\beta)(v_1, \cdots, v_k) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}(\text{sgn }\sigma)\beta(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

Note Alt $\beta = \beta$ if and only if β is alternating.

Similar to the symmetric case, the tensor product of two alternating tensors need not be alternating.

We instead introduce a new product called the alternating product.

For $\omega \in \bigwedge^k(\mathbb{V}^*)$ and $\eta \in \bigwedge^l(\mathbb{V}^*)$, define their alternating product to be

$$\omega \wedge \eta := \frac{(k+l)!}{k!l!}\text{Alt }(\omega \otimes \eta)$$

It is also called the wedge product. The numerical prefactor in the above is kept in order to allow simplifications for other expressions involving the wedge product.

We will deal exclusively with alternating tensors in a later section.

2 Some Differential Geometry

We will deal here exclusively with subsets(or more precisely,open submanifolds) of $\mathbb{R}^n$. We do not assume familiarity with theory of manifolds or differential geometry, although the experienced reader will find it easy to generalize the results to manifolds(with boundary).

Let $U \subset \mathbb{R}^n$ be open. We associate to each point $p \in U$, a vector space, which we call the *tangent space* of U at p, denoted T_pU. Geometrically, T_pU is the set of all "directions" at p, tangent to U, which we make precise as follows: let $\gamma : (-\varepsilon, \varepsilon) \to U$ be any smooth path for some $\varepsilon > 0$ with $\gamma(0) = p$, then we can associate to this curve the direction $\gamma'(0) \in \mathbb{R}^n$. We define T_pU to be the equivalence class of such paths, $[\gamma]$, under the relation $\gamma_1 \sim \gamma_2$ if $\gamma_1'(0) = \gamma_2'(0)$. Essentially any such equivalence class $[\gamma]$ represents the associated tangent direction $\gamma'(0)$. Observe, for any $v \in \mathbb{R}^n$, since U is open, there exists some $\varepsilon > 0$ such that $\gamma_v : (-\varepsilon, \varepsilon) \to U$, $\gamma_v(t) = p + tv$ is well defined. Hence, $[\gamma_v]$ corresponds to the direction $\gamma_v'(0) = v$. In fact the map $v \mapsto \gamma_v$ gives a bijection between $\mathbb{R}^n$ and T_pU. Hence, T_pU inherits the structure of a vector space under this bijection, such that it is linearly isomorphic to $\mathbb{R}^n$.

An alternative way to think of the tangent space is in terms of directional derivatives. We think of a tangent vector as a functional, acting on a (smooth) function to give the directional derivative in the direction of the tangent vector. The mathematical term for this is a *derivation*.

Let $C^\infty(U)$ denote the (vector) space of all smooth functions $f : U \to \mathbb{R}$ and $C_p^\infty(U)$ the set of all smooth functions $f : V \to \mathbb{R}$ where $V \subseteq U$ is an open subset containing p. The tangent space T_pU can be defined to be the set of all derivations $D : C_p^\infty(U) \to \mathbb{R}$, that is,

- D is linear, $D(af + bg) = aD(f) + bD(g)$ for all $f, g \in C_p^\infty(U)$ and $a, b \in \mathbb{R}$.
- D satisfies the Leibnitz rule, $D(fg) = f(p)D(g) + g(p)D(f)$ for all $f, g \in C_p^\infty(U)$.

One can check that the partial derivatives $\{\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \cdots, \frac{\partial}{\partial x^n}|_p\}$ are derivations and in fact form a basis for T_pU as per the above definition of the tangent space. Explicitly, $\frac{\partial}{\partial x^i}|_p$ acts on $f \in C_p^\infty(U)$ by

$$\frac{\partial}{\partial x^i}|_p(f) = \frac{\partial f}{\partial x^i}(p)$$

This is simply the directional derivative of f in the direction of the i-th standard basis vector. More generally, any vector $v \in T_pU$ can be written as

$$v = \sum_{i=1}^n v_i\frac{\partial}{\partial x^i}|_p$$

acting on $f \in C_p^\infty(U)$ as the directional derivative

$$v(f) = \langle(\nabla f), (v_1, \cdots, v_n)\rangle = \nabla_v f$$

We can piece together the tangent spaces at all points of U to form the *tangent bundle* TU, defined as

$$TU := \coprod_{p\in U}\{p\} \times T_pU$$

In general, the tangent bundle of a manifold is again a manifold, or more technically, a vector bundle, but in our simplified case, we can see $TU \cong U \times \mathbb{R}^n$. An element of the tangent bundle is usually denoted as (p, v) where $p \in U$, and $v \in T_pU$. We have a canonical projection $\pi : TU \to U$, given by $\pi(p, v) = p$ which gives the "source" of the tangent vector.

A (smooth) *vector field* on U is a smooth section of the tangent bundle, that is, a smooth map $X : U \to TU$ such that $\pi \circ X(p) = p$ for all $p \in U$. In simple words, a vector field X on U is just an assignment to each point p, a vector $X(p) \in T_pU$ in the tangent space of that point in a smooth manner. We often just write X_p for the value of the vector field at p. In terms of the standard basis of partial derivatives, we can write

$$X_p = \sum_{i=1}^{n} v^i(p) \frac{\partial}{\partial x^i}|_p$$

where $v^i : U \to \mathbb{R}$ are now smooth functions for $1 \leq i \leq n$.

The *cotangent bundle* is defined to be the dual of the tangent bundle. We associate to each point $p \in U$, the dual of the tangent space $(T_pU)^*$, and then patch them up together to get

$$T^*U = \coprod_{p \in U} \{p\} \times (T_pU)^*$$

The basis for $(T_pU)*$ dual to the standard basis of partial derivatives for T_pU is denoted by dx^i_p, that is $dx^i(\frac{\partial}{\partial x^j}|_p) = \delta^i_j$.

Let $V \subset \mathbb{R}^m$ be open. Given a smooth function $F : U \to V$, we define a map $dF_p : T_pU \to T_{F(p)}(V)$ by the following two equivalent ways(depending on how we view the tangent space):

- Given a vector $v \in T_pU$ that is the velocity vector of a curve $\gamma : (-\varepsilon, \varepsilon) \to U$, define $dF_p(v) = [F \circ \gamma] = (F \circ \gamma)'(0)$.
- Considering $v \in T_pU$ as a derivation, define $dF_p(v)$ to be the derivation whose action on $g \in C^\infty_{F(p)}(V)$ is given by $dF_p(v)(g) = v(g \circ F)$.

Stitching together the maps for all $p \in U$, we get the tangent map $dF : TU \to TV$. It is called the *differential* of F. This is a linear map and is in fact the total derivative of F. This is seen easily using the second definiton as follows: if $F = (F^1, \cdots, F^m)$ and $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}|_p$, then for $g \in C^\infty_{F(p)}(V)$, with coordinates y^j on V,

$$\begin{aligned} dF_p(v)(g) = v(g \circ F) &= \sum_{i=1}^{n} v^i \frac{\partial g \circ F}{\partial x^i}(p) \\ &= \sum_{i=1}^{n} v^i \sum_{j=1}^{m} \frac{\partial F(p)}{\partial x^i} \frac{\partial g(F(p))}{\partial y^j} \\ &= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} v^i \frac{\partial F(p)}{\partial x^i} \right) \frac{\partial}{\partial y^j}|_{f(p)} g \end{aligned}$$

so that the matrix of dF_p in the standard basis of partial derivatives is simply the Jacobian of F at p.

In the special case, where $f : U \to \mathbb{R}$, $df_p : T_pU \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ can be identified as an element of the cotangent space $(T_pU)^*$, with $df_p(v) := v(f)$ for $v \in T_pU$. In the standard dual basis,

$$df_p = \sum_{i=1}^{n} \frac{\partial f(p)}{\partial x^i} dx^i_p$$

3 Differential Forms

Recall that for a vector space $\mathbb{V}$, we have defined the space of alternating covariant k-tensors, $\bigwedge^k(\mathbb{V}^*)$. We examine these in more detail.

We first describe some notation. Given a positive integer k, a k-tuple of positive integers $(i_1, \cdots, i_k)$ is called a multi-index of length k. For $\sigma \in \mathfrak{S}_k$, denote by I_σ the multi-index $(i_{\sigma(1)}, \cdots, i_{\sigma(k)})$.

Let $\mathbb{V}$ be a n-dimensional vector space as usual, and let $\varepsilon^1, \cdots, \varepsilon^n$ be a basis for the dual space $\mathbb{V}^*$. For a multi-index $I = (i_1, \cdots, i_k)$, denote by ε^I the covariant k-tensor defined by

$$\varepsilon^I(v_1, \cdots, v_k) = \det \begin{pmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k) \end{pmatrix}$$

Note that ε^I is alternating, so $\varepsilon^I \in \bigwedge^k(\mathbb{V}^k)$. In fact one can check that $\varepsilon^I = \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}$.

For multi-indices I and J of length k, let

$$\delta^I_J = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \cdots & \delta^{i_k}_{j_k} \end{pmatrix} = \begin{cases} (\operatorname{sgn} \sigma) & \text{if } J = I_\sigma \\ 0 & \text{else} \end{cases}$$

Now, suppose that $\{\varepsilon^i\}$ is the dual basis to a basis $\{E_i\}$ for V. Then, from the above, $\varepsilon^I(E_{j_1}, \cdots, E_{j_k}) = \delta^I_J$ where $J = (j_1, \cdots, j_k)$.

A multi-index $I = (i_1, \cdots, i_k)$ is said to be increasing if $i_1 < \cdots < i_k$. A basis for $\bigwedge^k(\mathbb{V}^*)$ is given by

$$\{\varepsilon^I : I \text{ is an increasing multi-index of length } k\}$$

Hence, $\dim \bigwedge^k(\mathbb{V}^*) = \binom{n}{k}$. Note that there are no alternating tensors of rank greater than $n = \dim \mathbb{V}$ because for an alternating tensor $\alpha \in \bigwedge^k(\mathbb{V}^*)$, $\alpha(v_1, \cdots, v_k) = 0$ if the vectors $v_1, \cdots, v_k$ are linearly dependent, and any set of $n+1$ vectors is linearly dependent in $\mathbb{V}$. This follows from the fact that because of its alternating nature, the tensor changes sign whenever two arguments are switched, so if the arguments are equal it evaluates to zero.

In particular, the space $\bigwedge^n(\mathbb{V}^*)$ is one dimensional and spanned by $\varepsilon^1\wedge\varepsilon^2\wedge\cdots\wedge\varepsilon^n$. Its action on vectors is given by the determinant function. Hence, it is often called the volume form or orientation form.

We now list some properties of the wedge product which can be verified by straightforward calculations:

- $(\omega\wedge\eta)\wedge\xi = \omega\wedge(\eta\wedge\xi)$ for any alternating tensors ω,η,ξ. In other words, the wedge product is associative.

- The wedge product is not commutative but skew-commutative(or graded commutative), that is, for $\omega\in\bigwedge^k(\mathbb{V}^*)$ and $\eta\in\bigwedge^l(\mathbb{V}^*)$, we have
$$\omega\wedge\eta = (-1)^{kl}\eta\wedge\omega$$

- The wedge product is bilinear, that is
$$(a\omega+b\xi)\wedge\eta = a(\omega\wedge\eta)+b(\omega\wedge\xi)$$
for alternating tensors ω,ξ,η and $a,b\in\mathbb{R}$.

- For multi-indices $I=(i_1,\cdots,i_k)$ and $J=(j_1,\cdots,j_l)$, $\varepsilon^I\wedge\varepsilon^J=\varepsilon^{IJ}$ where $IJ=(i_1,\cdots,i_k,j_1,\cdots,j_l)$.

- For $\omega^1,\cdots,\omega^k\in\mathbb{V}^*$ and $v_1,\cdots,v_k\in\mathbb{V}$,
$$\omega^1\wedge\cdots\omega^k(v_1,\cdots,v_k) = \det\big(\omega_i(v^j)\big)$$

Define the space $\bigwedge(\mathbb{V}^*)=\bigoplus_{k=1}^n\bigwedge^k(\mathbb{V}^*)$, from the above this is a vector space of dimension 2^n. Under the wedge product, it becomes an associative algebra called the exterior algebra or Grassmann algebra of $\mathbb{V}$.

As in the case of symmetric algebra, we have an alternate description of the alternating algebra. Let I_{alt} be the ideal of $T(\mathbb{V}^*)$ generated by elements of the form $\alpha\otimes\alpha$ for $\alpha\in\mathbb{V}^*$. In other words I_{alt} is spanned by all those covariant tensors $\alpha_1\otimes\cdots\otimes\alpha_i\otimes\alpha_i\otimes\cdots\otimes\alpha_k$, that is, simple tensors with two(or more) factors equal. We define $\widetilde{\Lambda}(\mathbb{V}^*):=T(\mathbb{V}^*)/I_{alt}$. The tensor product descends to the quotient and forms the alternating product, that is, if $q:T(\mathbb{V}^*)\to\widetilde{\Lambda}(\mathbb{V}^*)$ is the quotient map, $\alpha\wedge\beta:=q(\alpha\otimes\beta)$. We can check that the map $\widetilde{A}:\widetilde{\Lambda}(\mathbb{V}^*)\to\bigwedge(\mathbb{V}^*)$ given by
$$\widetilde{A}(\alpha_1\wedge\cdots\wedge\alpha_k) := \sum_{\sigma\in\mathfrak{S}_k}(\operatorname{sgn}\sigma)\alpha_{\sigma(1)}\otimes\cdots\otimes\alpha_{\sigma(k)}$$
is an isomorphism of algebras. So, we can choose to think of the exterior algebra in either way.

Now, we can replace $\mathbb{V}$ with the tangent space T_pU. As in the case of the tangent and cotangent bundle, we can put the spaces $\bigwedge^k(T_pU^*)$ together for all points and define
$$\Lambda^kT^*U = \coprod_{p\in U}\{p\}\times\Lambda^k(T_pU^*)$$

As before, we have the projection $\pi:\Lambda^kT^*U\to U$. A *differential k-form* is a section of Λ^kT^*U, that is, a smooth map $\omega:U\to\Lambda^kT^*U$ satisfying $\pi\circ\omega(p)=p$. In other words, a differential k-form on U is an assignment to each point $p\in U$ an alternating k-tensor $\omega(p)\in\Lambda^k(T_pU^*)$. In terms of the standard basis $\{dx^i\}$ on T_pU^*, this involves giving a smooth function $\omega_I:U\to\mathbb{R}$ for each increasing multi-index of length k, so that we can write
$$\omega = \sum_{I:i_1<\cdots<i_k}\omega_I dx^I$$
or more explicitly,
$$\omega(p) = \sum_{I:i_1<\cdots<i_k}\omega_I(p)dx_p^{i_1}\wedge\cdots dx_p^{i_k}$$

We denote the vector space of smooth k-forms on U by $\Omega^k(U)$. Thus, 0-forms are just smooth functions on U and 1-forms are covector fields. Note that for any $\omega\in\Omega^k(U)$ and $f\in C^\infty(U)$, $f\omega=f\wedge\omega\in\Omega^k(U)$.

We now define an operation on differential forms, known as exterior differentiation. The exterior derivative d is a map $d:\Omega^k(U)\to\Omega^{k+1}(U)$. On 0-forms, we have already encountered it, it is nothing but the differential, $f\mapsto df$. This extends easily to k-forms, for $\omega=\omega=\sum_{I:i_1<\cdots<i_k}\omega_I dx^I\in\Omega^k(U)$), define
$$d\omega = \sum_{I:i_1<\cdots<i_k}(d\omega_I)\wedge dx^I$$
Explicitly,
$$d\omega = \sum_{I:i_1<\cdots<i_k}\sum_{i=1}^n\frac{\partial\omega_I}{\partial x^i}dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

For example, for a 1-form $\omega=\sum_{i=1}^n\omega_i dx^i$,
$$d\omega = \sum_{i<j}\left(\frac{\partial\omega_j}{\partial x^i}-\frac{\partial\omega_i}{\partial x^j}\right)dx^i\wedge dx^j$$

It follows from the equality of mixed partial derivatives that $d\circ d=0$. Furthermore d is a linear map satisfying the property
$$d(\omega\wedge\eta) = d\omega\wedge\eta+(-1)^k\omega\wedge d\eta$$
for $\omega\in\Omega^k(U)$ and $\eta\in\Omega^l(U)$.

We say a differential form ω is *closed* if $d\omega = 0$ and exact if $\omega = d\eta$ for some $(k-1)$-form η. Note that exact forms are always closed, since $d \circ d = 0$.

We end this section by defining the Hodge dual operator $\star : \bigwedge^k(\mathbb{V}^*) \to \bigwedge^{n-k}(\mathbb{V}^*)$ where $\dim \mathbb{V} = n$. It is a linear map and it is enough to define it on the basis vectors. For $\varepsilon^I = \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}$,

$$\star(\varepsilon^I) = \frac{1}{(n-k)!} \sum_{J=(j_1,\cdots,j_{n-k})} \epsilon_{i_1\cdots i_k j_1 \cdots j_{n-k}} \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_{n-k}}$$

where ϵ denotes the Levi-Civita symbol. This is in fact an isomorphism between alternating k-tensors and alternating $(n-k)$-tensors.

4 Div, Curl, Grad and Maxwell Equations

We specialise to the case $U \subset \mathbb{R}^3$. We know that electric and magnetic fields are described by vector fields on subsets of $\mathbb{R}^3$.

We label the co-ordinates x, y, z. A 0-form on U is just a function $f : U \to \mathbb{R}$, its exterior derivative is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

This is just the gradient of the function, ∇f.

Now, consider a 1-form $\omega = \omega_x dx + \omega_y dy + \omega_z dz$. Its exterior derivative is

$$d\omega = \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y}\right) dx \wedge dy + \left(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z}\right) dy \wedge dz + \left(\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x}\right) dz \wedge dx$$

We see that the components of $d\omega$ are just the components of the vector $\nabla \times \omega$ where $\omega = (\omega_x, \omega_y, \omega_z)$ by abuse of notation. More precisely, we can identify $\star(d\omega)$ with $\nabla \times \omega$ since $\star(dx \wedge dy) = dz$ and so on.

Finally, consider a 2-form $\eta = \eta_z dx \wedge dy + \eta_x dy \wedge dz + \eta_y dz \wedge dx$. Its exterior derivative is

$$d\eta = \left(\frac{\partial \eta_x}{\partial x} + \frac{\partial \eta_y}{\partial x} + \frac{\partial \eta_z}{\partial z}\right) dx \wedge dy \wedge dz$$

Hence, $\star(d\eta) = \nabla \cdot (\star\eta)$ where we write $\star\eta = (\eta_x, \eta_y, \eta_z)$.

Thus the vector operations of gradient, curl and divergence are just the exterior derivative in disguise. This also suggests that the various theorems linking these operations like the divergence theorem or Green's theorem are related in some manner. We do not pursue this further here.

We are now ready to present an elegant formulation of the Maxwell equations. For this part, we freely borrow notation used in physics namely the Maxwell field strength tensor $F_{\mu\nu}$ and hope that the reader is acquainted with four-vector notation from special relativity. Recall the four vector $A^\mu = (\phi, \bar{A})$ where ϕ is the electric scalar potential and $\bar{A}$ is the magnetic vector potential. We have $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

where E and B denote the electric and magnetic fields respectively.

Define the Faraday 2-form $F = -\frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$, one can check

$$\begin{aligned} F =& (E_x dx + E_y dy + E_z dz) \wedge dt \\ &+ B_z dx \wedge dy + B_x dy \wedge dz + B_y dz \wedge dx \end{aligned}$$

From this, note $\star F = -\frac{1}{2}F_{\mu\nu} \star (dx^\mu \wedge dx^\nu)$ is given by

$$\begin{aligned} \star F =& - E_x dy \wedge dz - E_y dz \wedge dx + E_z dx \wedge dy \\ &+ (B_x dx + B_y dy + B_z dz) \wedge dt \end{aligned}$$

If we define $G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}$, then it can be seen $\star F = -\frac{1}{2}G_{\mu\nu}dx^\mu \otimes dx^\nu$.

Finally define the current density 3-form J:

$$J = (j_x dy \wedge dz + j_y dz \wedge dx + j_z dx \wedge dy) \wedge dt - \rho dx \wedge dy \wedge dz$$

Then Maxwell's equations can be wriiten as

$$dF = 0, d \star F = J$$

5 Concluding Remarks

We have barely managed to touch the surface for the potential applications of differential forms not to mention the rich theory lying behind it. One of the major results we have merely hinted at here is the Stokes theorem, which unites various results like Gauss's divergence theorem, Green's theorem etc. under one umbrella. Undoubtedly, it is one of the most beautiful theorems with myriad uses.

We also note that using differential forms, one can detect "holes" in the manifold, readers are surely familiar with the differential form $\frac{ydx - xdy}{x^2+y^2}$ on $\mathbb{R}^2 \setminus 0$, which is closed but not exact. It detects the non simple connectedness of the domain. We encourage the reader to look at de Rham cohomology to get a better understanding of the above vague statements.

Finally, we remark that the tangent bundle introduced here is a specific example of the more general concept of vector bundle or fibre bundle. This is an essential concepts in areas like gauge theory. The reader is invited to look at the theory of fiber bundles in greater detail.